12.5. Continued

Prop Two planes are parallel if and only if their normal vectors are parallel

e.g. The planes $x+y+z=3$ and $2x+2y+2z=7$ are parallel

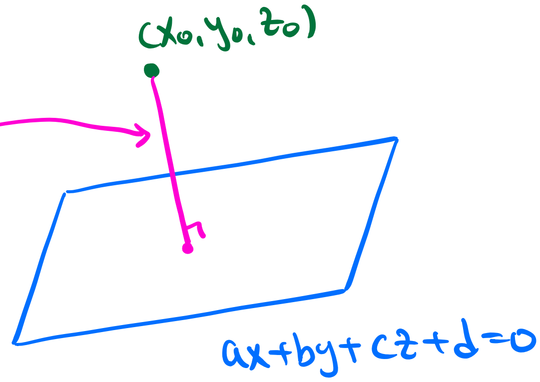
(normal vectors: $\vec{n_1}=(1,1,1)$, $\vec{n_2}=(2,2,2)$
$\rightsquigarrow$ $\vec{n_1}$ and $\vec{n_2}$ are parallel)

The planes $x-y+z=2$ and $3x-z=10$ are not parallel.

(normal vectors: $\vec{n_1}=(1,-1,1)$, $\vec{n_2}=(3,0,-1)$
$\rightsquigarrow$ $\vec{n_1}$ and $\vec{n_2}$ are not parallel)

Thm The distance from a point (x_0,y_0,z_0) to the plane $ax+by+cz+d=0$ is equal to

$$\boxed{\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}}$$



On the formula sheet.

Ex Find the two planes which are parallel to the plane $3x-2y+6z=0$ and 2 units away from it.

Sol The planes have a normal vector $\vec{n}=(3,-2,6)$

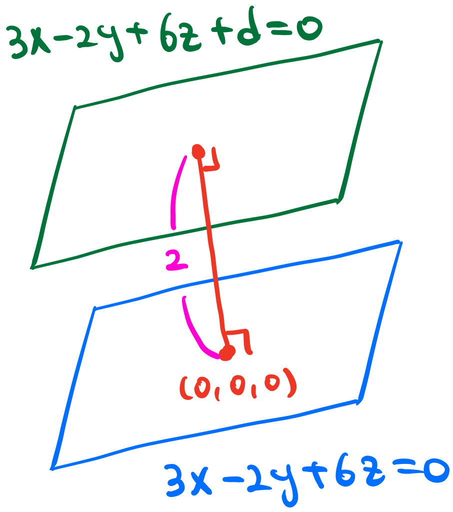
$\Rightarrow$ The plane equation must be

$3x-2y+6z+d=0$ for some d.

* To find the constant d, we choose a point on the given plane and consider the distance to the other plane.

Set $x=0, y=0$: $3x-2y+6z=0 \rightsquigarrow 6z=0 \rightsquigarrow z=0$.

$\Rightarrow (0,0,0)$ is on the plane $3x-2y+6z=0$.



Distance from $(0,0,0)$ to the plane $3x-2y+6z+d=0$ is 2.

$$\Rightarrow \frac{|3\cdot 0-2\cdot 0+6\cdot 0+d|}{\sqrt{3^2+(-2)^2+6^2}}=2.$$

$$\rightsquigarrow \frac{|d|}{7}=2 \rightsquigarrow |d|=14 \rightsquigarrow d=\pm 14.$$

$\Rightarrow$ The planes are $\boxed{3x-2y+6z\pm 14=0}$

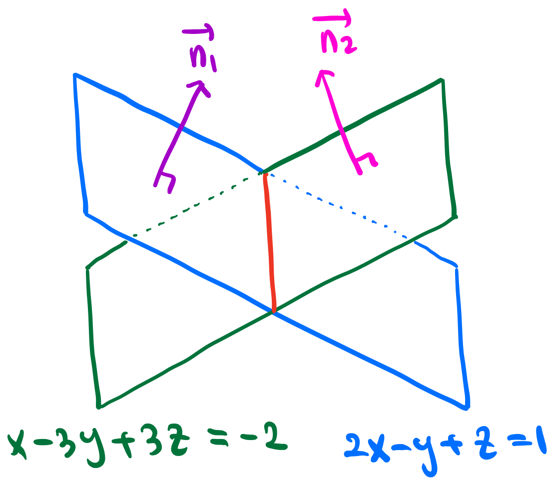
Ex Describe the intersection of the planes

$2x - y + z = 1$ and $x + 3y - 3z = 4$

Sol Normal vectors are $\vec{n_1} = (2, -1, 1)$, $\vec{n_2} = (1, 3, -3)$

$\Rightarrow$ The planes are not parallel.

$\Rightarrow$ The intersection is a line.



The line is perpendicular to both $\vec{n_1}$ and $\vec{n_2}$.

$\Rightarrow$ A direction vector is

$$\vec{n_1} \times \vec{n_2} = (2, -1, 1) \times (1, 3, -3)$$
$$= (0, 7, 7)$$

To find a point on the line, we solve the plane equations together by setting one of x, y, z to be 0.

* If the direction vector has a zero component, you should not set the corresponding variable to be 0.

$z = 0 \Rightarrow 2x - y = 1$ and $x + 3y = 4$ $\rightsquigarrow$ $x = 1$, $y = 1$.

$\Rightarrow (1, 1, 0)$ is on the line.

$\rightsquigarrow$ The intersection is $\boxed{\vec{l}(t) = (1 + 0 \cdot t,\ 1 + 7t,\ 0 + 7t)}$

Note On this line, the x-coordinate is always 1.

So you cannot find a point by setting $x = 0$.

12.6. Cylinders and quadric surfaces

Def (1) A quadric surface is a surface given by a degree 2 equation.

e.g. $x^2+y^2+z^2=1 \rightsquigarrow$ a sphere.

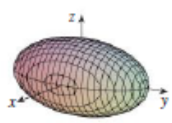
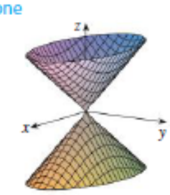
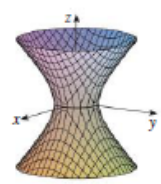
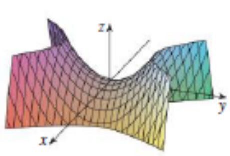
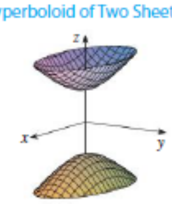
$x^2+y^2=4 \rightsquigarrow$ a circular cylinder.

$z=x^2+y^2 \rightsquigarrow$ a circular paraboloid.

(2) A cylinder is a surface made of parallel identical cross sections.

e.g. circular cylinders, prisms, ...

$*$ On the textbook, you can find a table of quadric surfaces with their equations and graphs.

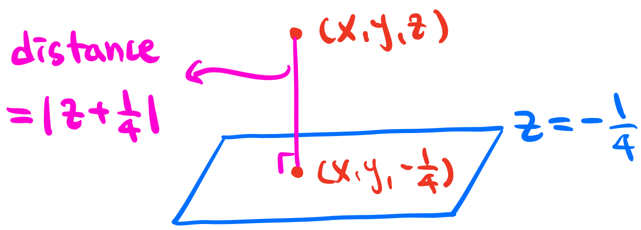
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Ex Describe each set using an equation and a graph.

(1) The set of all points which are equidistant from the point $(0,0,\frac{1}{4})$ and the plane $z=-\frac{1}{4}$.

Sol Distance from $(0,0,\frac{1}{4})$ is $\sqrt{x^2+y^2+(z-\frac{1}{4})^2}$

Distance from the plane $z=-\frac{1}{4}$ is $|z+\frac{1}{4}|$.

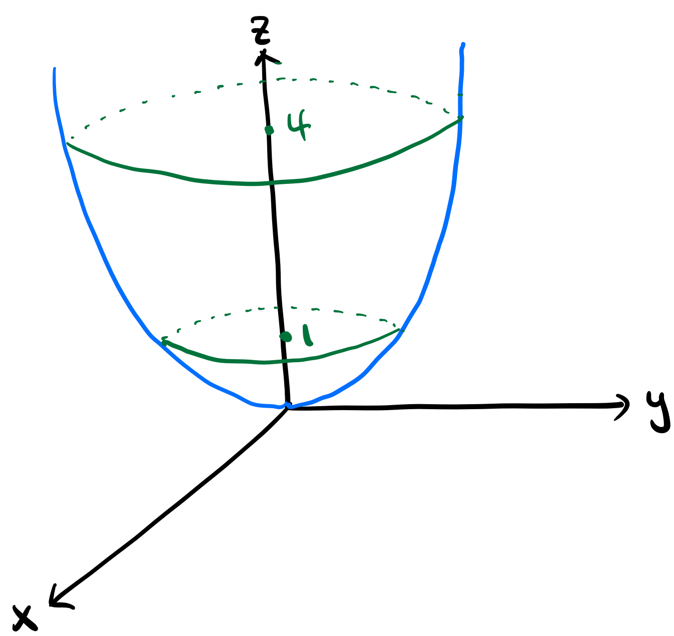


$$\Rightarrow \sqrt{x^2+y^2+(z-\tfrac{1}{4})^2} = |z+\tfrac{1}{4}|.$$

$$\rightsquigarrow x^2+y^2+(z-\tfrac{1}{4})^2 = (z+\tfrac{1}{4})^2.$$

$$\rightsquigarrow x^2+y^2+\cancel{z^2}-\tfrac{1}{2}z+\cancel{\tfrac{1}{16}} = \cancel{z^2}+\tfrac{1}{2}z+\cancel{\tfrac{1}{16}}$$

$$\rightsquigarrow \boxed{z=x^2+y^2}$$: a circular paraboloid



$x=0 \Rightarrow z=y^2$

$\rightsquigarrow$ a parabola

$z=1 \Rightarrow 1=x^2+y^2$

$\rightsquigarrow$ a circle

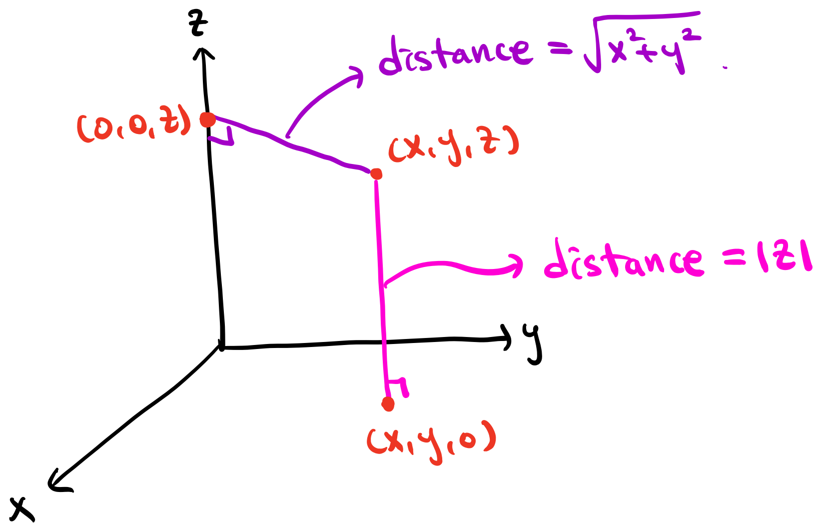
$z=4 \Rightarrow 4=x^2+y^2$

$\rightsquigarrow$ a circle

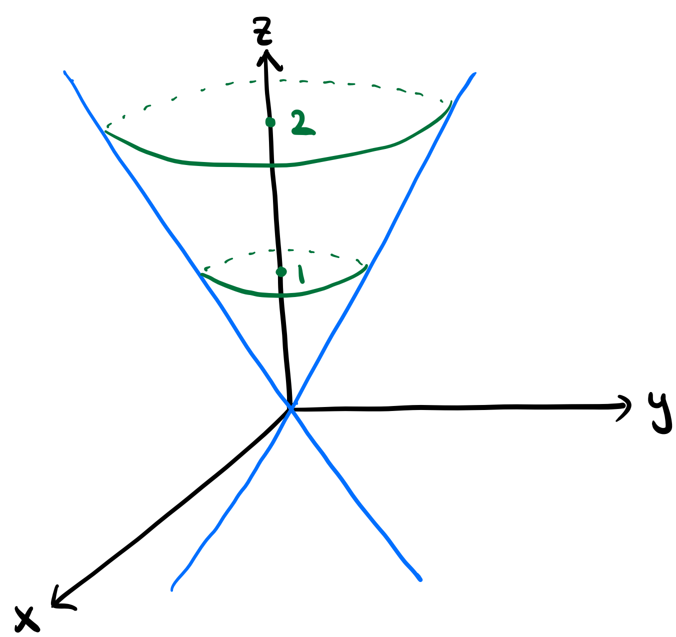
(2) The set of all points which are equidistant from the xy-plane and the z-axis.

Sol Distance from the xy-plane is $|z|$.

Distance from the z-axis is $\sqrt{x^2+y^2}$.



$\Rightarrow |z| = \sqrt{x^2+y^2} \leadsto \boxed{z^2 = x^2+y^2}$: a circular cone.



$x=0 \Rightarrow z^2 = y^2 \Rightarrow z = \pm y$

$\leadsto$ two lines

$z=1 \Rightarrow 1 = x^2+y^2$

$\leadsto$ a circle

$z=2 \Rightarrow 4 = x^2+y^2$

$\leadsto$ a circle